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Affine manifolds, lagrangian manifolds.

Abstract.

Let (M, ω) be a symplectic manifold endowed with a lagrangian foliation \mathcal{L} , it has been shown by Weinstein [16] that the symplectic structure of M defines on each leaf of \mathcal{L} , a connection which curvature and torsion forms vanish identically. Suppose that L_0 is a compact leaf which Weinstein connection is geodesically complete, Molino and Curras-Bosch [2] have classified germs of such lagrangian foliation around L_0 . In this paper we extend this classification without supposing the completeness of the compact leaf. The Weinstein connection is dual to the Bott connection, this enables to relate the conjecture of Auslander and Markus to transversally properties of these foliations.

1. The Weinstein connection.

In this section, we shall describe following Dazord [3], the Weinstein connection defined on the leaves of a lagrangian foliation. Let (M, ω) be a symplectic manifold endowed with a lagrangian foliation \mathcal{L} , L_0 a leaf of \mathcal{L} , $\chi(\mathcal{L})$ the vector fields tangent to \mathcal{L} and $\chi(n(\mathcal{L}))$ the sections of the normal bundle $n(\mathcal{L})$ of \mathcal{L} . One can define the Bott connection $\hat{\nabla} : \chi(\mathcal{L}) \otimes \chi(n(\mathcal{L}))$

$$\hat{\nabla}_X Y = u[X, Y'],$$

where X is an element of $\chi(\mathcal{L})$, Y' is an element of $\chi(M)$ over Y and u the canonical projection $\chi(M) \rightarrow \chi(n(\mathcal{L}))$. The connection $\hat{\nabla}$ induces a connection ∇' on $\chi(n(\mathcal{L})^*)$ defined by:

$$(\nabla'_X f)(t) = L_X(f(t)) - f(\hat{\nabla}_X t).$$

The symplectic duality allows to identify $\chi(\mathcal{L})$ to $\chi(n(\mathcal{L})^*)$, since \mathcal{L} is lagrangian, we deduce that the connection ∇' endows each leaf L_0 of \mathcal{L} with a connection ∇_{L_0} . It has been shown by Weinstein that the curvature and torsion forms of these connections vanish. This is equivalent to saying that the differentiable structure of each leaf of \mathcal{L} is defined by an atlas whose coordinates change are affine maps.

Consider a Darboux system of coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$ adapted to the foliation i.e. such that the foliation is defined by the equations $dq_1 = \dots = dq_n = 0$. The coordinates (p_1, \dots, p_n) define also the requested affine structure.

The connection ∇ is the dual of the Bott connection. This gives an isomorphism between the holonomy of ∇_{L_0} and the infinitesimal holonomy of L_0 . The purpose of this paper is to interpret properties and conjectures of affine manifolds theory in terms of the transverse geometry of lagrangian foliations and to study the neighbourhood of the compact leaf whose Weinstein connection is complete using the reduction principle, in Molino [9] and ideas of geometric non abelian higher cohomology as described in Tsemo [15].

2. The Auslander conjecture and the growth.

It has been conjectured by Auslander that the fundamental group of a compact and complete affine manifold is polycyclic. This is equivalent to say that the linear holonomy of the complete structure is polycyclic.

The Auslander conjecture have been studied by many authors: It has been proven in the following case:

- The dimension of M is less than 3, by Fried, and Goldman
- The Dimension of M is less than 6 by Abels, Soifer and Margulis
- The linear holonomy of the leaves is includes in $O(n-1, 1)$ by Goldman and Kamishima
- The linear holonomy is included in an algebraic group product of algebraic groups of rank less than 1 by Margulis,
- The affine automorphisms group is of codimension greater than 1 by Tsemo.

Definition 2.1.

Let \mathcal{L} be a foliation on a compact manifold M , and $(U_i)_{i \in I}$ a finite atlas of M such that the restriction of \mathcal{L} to each U_i is simple, consider a leaf L_0 of \mathcal{L} , and denote by P_i a connected component the intersection of U_i and L_0 . A path in L_0 associated to $(U_i)_{i \in I}$ is a family of open set (P_1, \dots, P_k) such that $P_i \cap P_{i+1}$ is not empty.

The growth map γ_{U_i} is the map such that $\gamma_{U_i}(n)$ is the number of P_j relatively to the atlas $(U_i)_{i \in I}$ which can be joined to P_i by a path which length is less than n . The growth of the L_0 is the growth of γ_{U_i} .

Suppose that M is compact, and L_0 is a compact leaf of \mathcal{L} , Haefliger has shown that there is a local transversal T to \mathcal{L} which intersects L_0 , a representation $\rho : \pi_1(L_0) \rightarrow \text{Diff}(T)$, such that the restriction of \mathcal{L} to a neighbourhood V of L_0 is the suspension foliation of the quotient of $\hat{L}_0 \times T$ by $\pi_1(L_0)$, where \hat{L}_0 is the universal cover of L_0 , and the action of $\pi_1(L_0)$ on $\hat{L}_0 \times T$ is given by the Deck transformations on \hat{L}_0 , and ρ on T .

Proposition 2.1.

Suppose that the leaf L_0 is compact, then the growth of a leaf L_1 in V (defined in the previous paragraph) as above is bounded by the growth of $\rho(\pi_1(L_0))$.

Proof.

The transversal can be chosen compact, we conclude using 1.29 of Godbillon [6].

Proposition 2.2.

Suppose that the growth of the leaves of \mathcal{L} in a neighbourhood of L_0 is polynomial, then the fundamental group of L_0 is polycyclic.

Proof.

Suppose that the fundamental group is not polycyclic, then the Tits alternative applied to the linear holonomy, implies that there exists a free subgroup of $\pi_1(L_0)$ generated by 2 elements, this implies that the infinitesimal holonomy and the holonomy of the leaf L_0 contains a free subgroup generated by 2 elements. the growth of all the leaves cannot be polycyclic. see also Plante and Thurston [12].

We can set the following conjecture which implies the Auslander conjecture:

Conjecture 2.3.

A compact affine manifold is complete if and only if it is a leaf of a lagrangian foliation of a compact symplectic manifold such that the growth of all the leaves are polynomial.

Remark.

There exists compact affine manifolds whose fundamental groups are not polycyclic, examples of such affine manifolds are the product of surfaces of genus greater than 2 by a circle.

Proposition 2.4.

Let (L_0, ∇_{L_0}) be a compact affine manifold whose linear holonomy is contained in $Gl(n, \mathbb{Z})$, then (L_0, ∇_{L_0}) is a leaf of a lagrangian foliation defined on a compact symplectic manifold.

Proof.

Let \hat{L}_0 be the universal cover of L_0 , the cotangent bundle T^*L_0 of L_0 is the suspension of \mathbb{R}^n over L_0 by the action

$$h : \pi_1(L_0) \rightarrow Gl(n, \mathbb{R}),$$

$$\gamma \rightarrow^t (L(h_{L_0})(\gamma^{-1})),$$

where $L(h_{L_0})$ is the linear holonomy of (L_0, ∇_{L_0}) . Since the image of $L(h_{L_0})$ is contained in $Gl(n, \mathbb{Z})$, the image of h is also contained in $Gl(n, \mathbb{Z})$. Consider now translations t_{e_1}, \dots, t_{e_n} of \mathbb{R}^n , where e_1, \dots, e_n is a basis of \mathbb{R}^n . The quotient (M, ∇_M) of T^*L_0 fiber by fiber by the group generated by t_{e_1}, \dots, t_{e_n} is a compact affine manifold which is a suspension of the torus over L_0 . Let D be the developing map of L_0 , then the canonical symplectic form of \mathbb{R}^{2n} such that the affine space $x \times \mathbb{R}^n$, an the open set of affine spaces $D(\hat{L}_0) \times y$ are lagrangian is preserved by the holonomy of (M, ∇_M) . This implies that the pulls back to \hat{M} of this symplectic form defines on (M, ∇_M) a symplectic affine manifold, and the suspension foliation is lagrangian.

Let (M, ω) be a compact symplectic manifold endowed with a lagrangian foliation \mathcal{L} , which has a compact and complete leaf (L_0, ∇_{L_0}) . Weinstein has shown that there exists a neighbourhood U of L_0 which is symplectomorph to a

neighbourhood of the 0 section of the cotangent bundle of L_0 . The infinitesimal holonomy of the restriction of the horizontal foliation (if (x_1, \dots, x_n) is a system of affine coordinates of L_0 , then the horizontal foliation is defined by $(dx_1 = \dots = dx_n = 0)$ of T^*L_0 to U , and the one of \mathcal{L} coincide, we deduce that the Auslander conjecture is true if the combinatoric growth of the horizontal foliation of T^*L_0 is polynomial. To generalize proposition 2.4, one may also find cocompact affine action of T^*L_0 .

Milnor [8] asked whether the fundamental group of a complete affine manifold is polycyclic. Margulis [7] has constructed an example of a compact and complete affine manifold whose fundamental group is a free group generated by 2 elements. Other examples of free groups which are fundamental groups of complete affine manifolds have been constructed by Goldman, Charette and Drumm [1]. One may ask whether those manifolds are leaves of lagrangian foliations defined on compact symplectic manifolds.

If a compact affine manifold L_0 is not complete, then its fundamental group may not be polycyclic, it is the case of the product of a surface whose genus is greater than 2 by a circle. Are such manifolds leaves of lagrangian foliations on compact manifolds?

Examples of compact symplectic manifolds endowed with lagrangian foliations which has a compact leaf.

We will define first two lagrangian foliations on the torus T^4 endowed with a compact leaf.

The first example is the flat riemannian structure, we consider the quotient of \mathbb{R}^4 by the translations t_{e_1}, \dots, t_{e_4} . The symplectic structure is the one induced by the one of \mathbb{R}^4 defined by the form $\omega_0 = dx_1 \wedge dx_3 + dx_2 \wedge dx_4$. The lagrangian foliation is the foliation by torus which are the projection by the cover map of the affine spaces parallel to $\text{vect}(e_1, e_2)$.

The second example is the quotient of \mathbb{R}^4 by the transformations:

$$h_1(x_1, x_2, x_3, x_4) = (x_1 + 1, x_2, x_3, x_4)$$

$$h_2(x_1, x_2, x_3, x_4) = (x_1 + x_2, x_2 + 1, x_3, -x_3 + x_4)$$

$$h_3(x_1, x_2, x_3, x_4) = (x_1, x_2, x_3, x_4 + 1)$$

$$h_4(x_1, x_2, x_3, x_4) = (x_1 + x_2, x_2, x_3 + 1, -x_3 + x_4)$$

The symplectic structure is the projection of ω_0 . The lagrangian foliation here is defined by the projection of the affine space parallel to $\text{vect}(e_1, e_2)$. The compact leaf is the projection of the affine space parallel to $\text{vect}(e_1, e_2)$ which contains 0.

We consider the quotient of \mathbb{R}^4 , by h_1, h_2, t_{e_3} and t_{e_4} , it is a symplectic manifold endowed with the projection of ω_0 , the quotient of the affine space parallel to $\text{vect}(e_1, e_2)$ is the lagrangian foliation.

3. Markus Conjecture and transversely measurable foliations.

Markus has conjectured that a compact affine manifold is complete if and only if it is unimodular. Unimodular means that its linear holonomy group is a subgroup of $Sl(n, \mathbb{R})$.

Suppose that the compact and unimodular affine manifold (L_0, ∇_{L_0}) is a leaf of a lagrangian foliation \mathcal{L} defined on the compact manifold M . Let V be the neighbourhood of L_0 , on which \mathcal{L} is the quotient of $\hat{L}_0 \times T$, by the action of $\pi_1(M)$, where T is a closed lagrangian transversal to \mathcal{L} , $\pi_1(M)$ acts on \hat{L}_0 by the deck transformations, and on T by the holonomy representation of the foliation. Since the symplectic duality defines an isomorphism between the infinitesimal holonomy of \mathcal{L} and the linear holonomy of the affine structure of L_0 , we deduce that the holonomy of \mathcal{L} in L_0 preserves a measure on T . This leads to the following conjecture which implies the Markus conjecture:

Conjecture 3.1.

A compact affine manifold L_0 is complete if and only if it is the leaf of a lagrangian foliation which admits a transverse measures which support contains the saturated set of a transversal to L_0 .

It has been shown by Plante that the growth of a leaf contained in the support of a transverse measure of a codimension 1 foliation on a compact manifold is polynomial. This result is not true for codimension greater than 1 as shows certain suspensions of S^2 .

A complete affine manifold which fundamental group is polycyclic is unimodular, one may ask if a compact and complete affine manifold is a leaf of a lagrangian foliation on a compact manifold which admits a transverse measure which support contains a saturated space of a transversal to the compact leaf then the growth of the leaves are polynomial?

Proposition 3.2.

Let (M, ∇_M, ω_M) be a symplectic affine manifold, that is an affine manifold (M, ∇_M) endowed with a symplectic form ω_M parallel respectively to ∇_M . Suppose that there exists in M an affine lagrangian foliation which has a compact leaf L_0 , and such that the fundamental group of L_0 is normal in $\pi_1(M)$, then M is the quotient of the cotangent bundle of L_0 by a discrete group of affine symplectomorphisms.

Proof.

Let x_0 be an element of L_0 , the Weinstein connection is the restriction of the connection of (M, ∇_M) to L_0 . Let \mathbb{R}^{2n} be the universal covering space of M , endowed with the pulls back of the symplectic form of M . The Dazord theorem implies that the quotient of \mathbb{R}^{2n} by $\pi_1(L_0)$ is the cotangent bundle of L_0 . Since

we have supposed that $\pi_1(L_0)$ is normal in $\pi_1(M)$, we conclude that (M, ∇_M) is the quotient of T^*L_0 by the cocompact group $\pi_1(M)/\pi_1(L_0)$.

4. Classification of symplectic manifolds endowed with a lagrangian foliation.

Let (M, ω) be a symplectic manifold endowed with a lagrangian foliation \mathcal{L} which has a compact leaf L_0 , suppose that there exists a parallel action respectively to the Weinstein connection of the torus T^1 on L_0 , Molino [9] has extended the action of S^1 to M , and constructed a Marsden-Weinstein reduction (M_1, ω_1) endowed with a Lagrangian foliation \mathcal{L}_1 which has a compact leaf L_1 which is the basic space of an affine bundle which total space is L_0 .

Suppose that the leaf L_0 is diffeomorphic to the n -dimensional torus, then there exists a left symmetric algebra (associative since commutative) \mathcal{H} , which defines a left complete invariant affine structure on the commutative group H such that L_0 is the quotient of H by a lattice $\pi_1(L_0)$. The Lie group H is also the connected component of the group of affine automorphisms of (L_0, ∇_{L_0}) , under this identification, the associative product of \mathcal{H} if the restriction of the associative product of $aff(\mathbb{R}^n)$ which is defined by

$$(C, c)(D, d) = (CD, C(d))$$

where C, D are elements of $gl(n, \mathbb{R})$ and c, d are elements of \mathbb{R}^n .

Given generators, $\gamma_1, \dots, \gamma_n$ of $\pi_1(L_0)$, there exists elements $(C_1, c_1), \dots, (C_n, c_n)$ such that $\gamma_i = \exp((C_i, c_i))$, we will say that the associative structure is rational if and only if the \mathbb{Q} -vector space generated by $(C_1, c_1), \dots, (C_n, c_n)$ is stable by the associative structure.

Proposition 4.1.

Let M be a complete affine manifold diffeomorphic to the torus, suppose that the associative algebra which defines its affine structure is rational, then there exists a parallel action of T^1 on M .

Proof.

Let (C, c) be an element of the \mathbb{Q} -vector space generated by $(C_1, c_1), \dots, (C_n, c_n)$ such that $(C, c).(C, c) = (C^2, C(c)) = 0$. Let $\gamma_1, \dots, \gamma_p$ be generators of $\pi_1(M)$ which defines the rational structure, with $\gamma_i = \exp((C_i, c_i))$. We have $(C, c) = \alpha_1(C_1, c_1) + \dots + \alpha_p(C_p, c_p)$, since \mathcal{H} is commutative, where α_i is a rational. There exists an integer p , such that $p\alpha_i$ is an integer, this implies that $\exp(pu) \in \pi_1(M)$, we conclude using Tsemo [14].

Corollary 4.2.

Let (M, ω) be a symplectic manifold endowed with a lagrangian foliation \mathcal{L} which has a compact leaf L_0 which affine structure is isomorphic to the rational affine structure of a complete torus, then there exists a sequence $(M_1, \omega_1), \dots, (M_1, \omega_1)$ of symplectic manifolds such that (M_1, ω_1) is (M, ω) , (M_1, ω_1) is the two dimensional torus, and (M_i, ω_i) is the Marsden-Weinstein reduction of (M_{i+1}, ω_{i+1}) acted on by the circle.

Proof.

Let \mathcal{H} be the rational associative commutative subalgebra \mathcal{H} of $aff(\mathbb{R}^n)$ which defines the affine structure of \mathcal{L}_0 .

We have seen that there exists a translation $t_u \in \pi_1(L_0)$, the quotient of L_0 by t_u is a $n - 1$ -dimensional torus L_1 , which affine structure is defined by the quotient \mathcal{H}_1 of the associative algebra \mathcal{H} by the Lie algebra of t_u . Since \mathcal{H} is rational, \mathcal{H}_1 is rational. Then if one performs a Marsden-Weinstein reduction on a M acted on by the extended action of t_u , he obtains a symplectic manifold M_1 which has a lagrangian foliation which has a compact leaf L_1 acted on by a parallel action of the circle. We can continue this procedure until a torus.

There are complete affine structures on the 2-dimensional torus T^2 , which holonomy group does not contain a translation.

Consider the 2-dimensional commutative Lie subgroup H [4], of $Aff(\mathbb{R}^2)$ which elements are $f_{s,t}(x, y) = (x + sy + \frac{s^2}{2} + t, y + s)$. This group acts simply transitively on \mathbb{R}^2 . Let h be a non rational real number, the subgroup I of H generated by $f_{h,0}$ and $f_{1,1}$ is a lattice of H . The quotient of H by I is a complete affine torus which holonomy does not have a translation.

More generally, Let $(L_n, \nabla_{L_n}) \rightarrow \dots \rightarrow (L_1, \nabla_{L_1})$ be a sequence of compact and complete affine manifolds where $(L_{i+1}, \nabla_{L_{i+1}}) \rightarrow (L_i, \nabla_{L_i})$ is an affine bundle which fiber is the circle such that $(L_{i+1}, \nabla_{L_{i+1}})$ is a leaf of a lagrangian foliation defined on the symplectic manifold M_{i+1} , and the action of the circle on L_{i+1} extends to an hamiltonian action on M_{i+1} such that the Marsden-Weinstein reduction of M_{i+1} is the symplectic manifold M_i endowed with a lagrangian foliation which has as compact leaf L_i . In this part we will classify the sequence germs of neighbourhood of L_i in M_i for such sequence of symplectic manifolds $M_n \rightarrow \dots \rightarrow M_1$ for a given sequence of affine manifolds $(L_n, \nabla_{L_n}) \rightarrow \dots \rightarrow (L_1, \nabla_{L_1})$.

The classification of the neighbourhood of a compact leaf of a lagrangian foliation.

In this part we recall the classication did by Molino and Curras-Bosch that we will generalize.

Consider a compact and complete n -dimensional affine manifold L_0 , x_0 an element of L_0 , and $h_{\nabla_{L_0}}$ the linear holonomy of L_0 , we identify $T_{x_0}L_0$ to $T_0^*\mathbb{R}^n$ endowed with its canonical flat connection. Let $h_{\nabla_{L_0}}$ be the holonomy representation. Its linear part induces a representation $h'_{\nabla_{L_0}} : \pi_1(L_0) \rightarrow T_0^*\mathbb{R}^n$. Suppose also defined $h_{x_0} : \pi_1(L_0) \rightarrow Diff(\mathbb{R}^n)$, a representation. The homomorphism h_{x_0} is the holonomy of a lagrangian foliation which has L_0 as a compact leaf if and only if for every element of D_0 the set of differentiable functions f defined in a neighbourhood of 0 in \mathbb{R}^n such that $f(0) = 0$ we have:

$$(1) \quad h_{\nabla_{L_0}}(\gamma) \circ d_0 = d_0 \circ h_{x_0}(\gamma^{-1})^*,$$

where $h_{x_0}^*(\gamma)(f) = f \circ h_{x_0}(\gamma)$.

The representation $h'_{\nabla_{L_0}}$ endows $T^*\mathbb{R}^n$ with a $\pi_1(L_0)$ module, we will denote by $H^*(\pi_1(L_0), T^*\mathbb{R}^n)$ the corresponding cohomology modules. The radiance obstruction map $\gamma \rightarrow h_{\nabla_{L_0}}(0)$ defines an element $[h'_{\nabla_{L_0}}]$ of $H^1(\pi_1(L_0), T^*\mathbb{R}^n)$.

The representation h_{x_0} endows D_0 with a module structure defined by

$$\gamma \circ f = f \circ h_{x_0}(\gamma^{-1})$$

The map d_0 induces a morphism

$$d_0^* : H^1(\pi_1(L_0), D_0) \rightarrow H^1(\pi_1(L_0), T^*\mathbb{R}^n)$$

Theorem [2] 4.3.

The germs of lagrangian foliations in a neighbourhood of (L_0, ∇_{L_0}) which foliation holonomy is h_{x_0} are classified up to symplectomorphisms by the elements of $d_0^{-1}[h_{\nabla_{L_0}}]$.*

Classification of germs of lagrangians foliation around a compact leaf L_0 which Weinstein connection is not necessarily complete.

In this paragraph, we will generalize the classification of Curras-Bosch and Molino without supposing that (L_0, ∇_{L_0}) is a complete affine manifold.

We can identify a neighbourhood of L_0 to a neighbourhood of the trivial section of T^*L_0 , we can extend the developing map $D_0 : \hat{L}_0 \rightarrow \mathbb{R}^n$ to a map $D : T^*\hat{L}_0 \rightarrow \mathbb{R}^{2n}$ such that for each element (x, y) of T^*L_0 , we have $D(x, y) = (D_0(x), y)$. This definition is possible since $T^*\hat{L}_0$ is a trivial bundle.

We consider a transversal T diffeomorphic to a contractible open set of the fiber of x_0 . The lagrangian foliation in U_0 is defined by a suspension, its lift to the universal cover \hat{U}_0 is the the the trivial foliation of $\hat{L}_0 \times T$. For each element x of \hat{L}_0 , the symplectic duality allows to identify $T^*(x \times T)$ to $T\hat{L}_{0,x}$, we can also identify $T^*(x \times T)$ to $T^*(x_0 \times T)$. Let U_x be a neighbourhood of 0 in $T\hat{L}_{0,x}$ such that the restriction of exp_x associated to the affine connection to it is injective. We can define the chart $\phi_x = exp_x : U_x \rightarrow \hat{L}_0$, the symplectic duality allows to identify U_x to a an open set V_x which contains 0 in $T^*(x \times T) = T^*(x_0 \times T)$ defines a chart $V_x \rightarrow L_0$. Following the well-known construction of the developing map, we can define a developing map $D_0 : \hat{L}_0 \rightarrow T^*T_{x_0}$.

We have:

$$(1) \quad h_{\nabla_{L_0}}(\gamma) \circ d_{x_0} = d_{x_0} \circ h_{x_0}(\gamma^{-1})^*,$$

where $h_{x_0}^*(\gamma)(f) = f \circ h_{x_0}(\gamma)$.

Conversely, let (L_0, ∇_{L_0}) be a compact affine manifold, and $h_{x_0} : \pi_1(L_0) \rightarrow Diff(\mathbb{R}^n)$ a symplectic form. Endows \mathbb{R}^{2n} with a symplectic form which pulls-back by the developing map of T^*L_0 is the pulls back to $T^*\hat{L}_0$ of the symplectic

form of T^*L_0 . As above, we can identify using the symplectic duality the developing map of (L_0, ∇_{L_0}) to a map $\hat{L}_0 \rightarrow T_0^*\mathbb{R}^n$.

The fact that the holonomy of L_0 preserve the pulls-back of this symplectic form to $\mathbb{R}^n \times \hat{L}_0$ means that:

$$(1) \quad h_{\nabla_{L_0}}(\gamma) \circ d_0 = d_0 \circ h_{x_0}(\gamma^{-1})^*,$$

we thus deduce a map

$$d_0^* : H^1(\pi_1(L_0), D_0) \rightarrow H^1(\pi_1(L_0), T_0^*\mathbb{R}^n)$$

We have the classification theorem

Theorem 4.4.

The germs of lagrangian foliations in a neighbourhood of (L_0, ∇_{L_0}) which foliation holonomy is h_{x_0} are classified up to a symplectomorphism by the elements of $d_0^{-1}[h_{\nabla_{L_0}}]$.*

Now we classify the sequence mentioned at the beginning. The affine manifolds (L_i, ∇_{L_i}) will be supposed to be compact and complete. Given an isomorphism class e_1 of a germ of a lagrangian foliation in a neighbourhood of (L_1, ∇_{L_1}) we will classify the germs of neighbourhoods of lagrangian foliations which has (L_2, ∇_{L_2}) has a leaf, and such that the Marsden-Weinstein reduction of a neighbourhood U_2 (related to the circle parallel action) of L_2 is a symplectic manifold (M_1, ω_1) endowed with a lagrangian foliation which has (L_1, ∇_{L_1}) has compact leaf, and which neighbourhood is isomorphic to e_1 .

Let x_2 be an element of L_2 which image by the canonical projection $p_2 : L_2 \rightarrow L_1$ is the element x_1 of L_1 . A germ of a lagrangian foliation on U_2 which has (L_2, ∇_{L_2}) has a leaf is defined by the following data:

A representation $h_{x_2} : \pi_1(L_2) \rightarrow \text{Diff}(\mathbb{R}^{n+1})$, which satisfies the relation (1),

Suppose that the Marsden-Weinstein reduction of U_2 is a symplectic manifold (M_1, ω_1) endowed with a lagrangian foliation which has L_1 as a leaf, and which neighbourhood is e_1 , then we have:

Proposition.

The following square is commutative:

$$(2) \quad \begin{array}{ccc} \pi_1(L_2) & \xrightarrow{h_{x_2}} & \text{Diff}(\mathbb{R}^{n+1}) \\ \downarrow p_2 & & \downarrow \\ \pi_1(L_1) & \xrightarrow{h_{x_1}} & \text{Diff}(\mathbb{R}^n) \end{array}$$

Proof.

Consider the moment map $J : U_2 \rightarrow \mathbb{R}$, we have supposed that $J^{-1}(0)$ contains L_2 . The transversal T_2 to the lagrangian foliation is diffeomorphic to an open set of \mathbb{R}^{l_1+1} , it is enough to remark that we can assume that the moment map is a coordinate function in a neighbourhood of 0. To do this, the function used to extend the action of the circle in a neighbourhood of L_2 see Molino Theorem 3.2 p. 186 must depend only of one coordinate in a neighbourhood of 0, and be equal to this coordinate in this neighbourhood of 0, since the basic function associated coincide with the moment map in this neighbourhood.

We will denote by e_2 the isomorphism class of this germ. We will endow the set (e_1, e_2) where e_1 is fixed with a structure of a gerbe.

Let Et_{L_0} be the site which objects are covering spaces of L_0 . To each object e of Et_{L_0} , we define the category $\text{symp}(e, L_0)$ of germs of lagrangian foliations which has e as a leaf, and such that the classical holonomy of an element of $\text{symp}(e, L_0)$ in x_e , the element of e over x_0 is the restriction of h_{x_0} to $\pi_1(e)$. The automorphisms of objects of $\text{symp}(e, L_0)$ are exponentials of hamiltonians flows defined by basic functions of the lagrangian foliation. It is easy to check that the correspondence $e \rightarrow \text{symp}(e, L_0)$ defines a gerbe on Et_{L_0} . That is the following axioms are satisfied:

- For each map $U \rightarrow V$, between objects of Et_{L_0} we have a pulls-back map $r_{U,V} : \text{symp}(V, L_0) \rightarrow \text{symp}(U, L_0)$ such that $r_{U,V} \circ r_{V,W} = r_{U,W}$.
- Gluing conditions for objects,

Consider a covering family $(U_i)_{i \in I}$ of an open set U of M , and for each i , an object x_i of $\text{symp}(U_i, L_0)$. Suppose that there exists a map $g_{ij} : r_{U_i \cap U_j, U_j}(x_j) \rightarrow r_{U_i \cap U_j, U_i}(x_i)$ such that $g_{ij}g_{jk} = g_{ik}$, then there exists an object x of $\text{symp}(U, L_0)$ such that $r_{U_i, U}(x) = x_i$.

Gluing conditions for arrows,

Consider two objects P and Q of $\text{symp}(L_0, L_0)$, the map $U \rightarrow \text{Hom}(r_{U,M}(P), r_{U,M}(Q))$ is a sheaf.

- There exists a covering family $(U_i)_{i \in I}$ of Et_{L_0} such that for each i the category $\text{symp}(U_i, L_0)$ is not empty.

- Let U be an object of Et_{L_0} , for each objects x and y of $\text{symp}(U, L_0)$, there exists a covering family $(U_i)_{i \in I}$ of U such that $r_{U_i, U}(x)$ and $r_{U_i, U}(y)$ are isomorphic.

Every arrow of $\text{symp}(U, L_0)$ is invertible, and there exists a sheaf Ham in groups on M , such that for each object x of $\text{symp}(U, L_0)$, $\text{Hom}(x, x) = \text{Ham}(U)$, and the elements of this family of isomorphisms commute with the restriction maps.

Remark.

The gerbe $\text{symp}(L_0)$ that we have just defined is a trivial gerbe as shows the previous classification theorem. We will denote by $\text{Symp}(L_0)$.

Let f be a global section of $\text{symp}(e, L_0)$, L_{0f} the pulls-back of L_0 to f , i.e the leaf of f which projects to L_0 . Consider a lagrangian 1-connected transversal T of \mathcal{L}_f to L_{0f} , the symplectic duality allows us to identify f to the vertical foliation of $T_{x_0}^* T$, the automorphisms of this objects are vertical homomorphisms

of the foliation. This group is a commutative group since T is lagrangian, and its elements are exponential of vertical hamiltonian vector fields.

The general case.

Consider a sequence $(L_l, \nabla_{L_l}) \rightarrow \dots \rightarrow (L_1, \nabla_{L_1})$ a sequence of compact and complete affine manifolds, such that the map $f_i : (L_{i+1}, \nabla_{L_{i+1}}) \rightarrow (L_i, \nabla_{L_i})$ is an affine bundle which typical fiber is the circle, our purpose is to classify sequences of germs, of lagrangian foliations U_i such that $(L_{i+1}, \nabla_{L_{i+1}})$ is a compact leaf of U_{i+1} , and L_i is a compact leaf of the lagrangian foliation of the Weinstein-Marsden reduction (M_i, ω_i) of U_{i+1} by the hamiltonian action of the circle.

The germs of the lagrangian foliation of U_i is defined by a representation

$$\pi_1(L_i) \rightarrow \text{Diff}(\mathbb{R}^n)$$

which satisfies condition (1), such that the following square is commutative:

$$\begin{array}{ccc} \pi_1(L_{i+1}) & \xrightarrow{h_{x_{i+1}}} & \text{Diff}(\mathbb{R}^{n+1}) \\ \downarrow p_i & & \downarrow \\ \pi_1(L_i) & \xrightarrow{h_{x_i}} & \text{Diff}(\mathbb{R}^n) \end{array}$$

Let e_1 be an object of $\text{symp}(L_1)$, we will associate to e_1 , the gerbe $\text{symp}(L_2, e_1)$ which is a gerbe defined on the site Et_{L_2} , such that for each object e_2 of Et_{L_2} , the objects of the category $\text{symp}(L_2, e_1)(e_2)$ are germs U_2 of lagrangians foliations which has L_2 as a leaf, and such that the Marsden-Weinstein reduction of U_2 by the hamiltonian action of the circle is a manifold endowed with a lagrangian foliation which has L_1 as a leaf and is isomorphic to e_1 .

Suppose defined the gerbe $\text{symp}(L_p, e_1, \dots, e_{p-1})$, and let e_p be an object of this gerbe. We will define the gerbe $\text{symp}(L_{p+1}, \dots, e_p)$ as the category which is a gerbe defined on the site $\text{Et}_{L_{p+1}}$, the objects of the category $\text{symp}(L_{p+1}, \dots, e_p)$ are germs U_{p+1} of lagrangian foliations which has L_{p+1} as leaf, and such that the Marsden-Weinstein reduction of U_{p+1} by the hamiltonian action of the circle is a manifold endowed with a lagrangian foliation which has L_p as a leaf, and such that the germ of this foliation around L_p is e_p .

The gerbes $\text{symp}(L_{p+1}, \dots, e_p)$ are trivial gerbes. For objects e_p and e'_p of $\text{symp}(L_p, \dots, e_{p-1})$, and a morphism $f : e_p \rightarrow e'_p$, there is a functor, $f^* : \text{symp}(L_{p+1}, \dots, e'_p) \rightarrow \text{symp}(L_{p+1}, \dots, e_p)$ such that there exists an isomorphism:

$$c(f, g) : (fg)^* \rightarrow g^* f^*$$

which satisfies a 1-descent condition that is:

$$(Id * c(f, g) \circ c(fg, h) = c(g, h) * Id \circ c(f, gh)$$

Suppose that the dimension of L_1 is l_1 , then the dimension of L_i is $l_i + i - 1$. We will denote by D_i the germs at 0 of differentiable functions of $\mathbb{R}^{l_i + i - 1}$ which

take the value 0 at the origin. The holonomy in x_i endows D_i with the structure of a $\pi_1(L_i)$ module by setting:

$$\gamma \circ f = f \circ h_{x_i}(\gamma^{-1})$$

the differential at the origin d_i induces a morphism:

$$d_i^* : H^1(\pi_1(L_i), D_i) \longrightarrow H^1(\pi_1(L_i), T_0^* \mathbb{R}^{l_1+i-1})$$

The isomorphism classes of germs of lagrangian foliation for which the holonomy is h_{x_i} are the elements of $d_i^{*-1}[h_{\nabla_{L_i}}]$.

Given a global section e_i of $\text{symp}(L_i)$, classified by the class $[c_i]$ of $H^1(\pi_1(L_i), D_i)$, we will determine the the classifying cocycle of the global sections of $\text{symp}(L_{i+1})$ which gives rise to e_i .

The surjection $p_i : \pi_1(L_{i+1}) \rightarrow \pi_1(L_i)$, and the commutative square (1) induces a the following commutative square:

$$\begin{array}{ccc} H^1(\pi_1(L_{i+1}), D_{i+1}) & \longrightarrow & H^1(\pi_1(L_i), \mathbb{R}^{l_1+i}) \\ f_i \downarrow & & g_i \downarrow \\ H^1(\pi_1(L_i), D_i) & \longrightarrow & H^1(\pi_1(L_i), \mathbb{R}^{l_1+i-1}) \end{array}.$$

The image of the classifying cocycle of elements of $\text{symp}(L_{i+1})$ by f_i is the classifying cocycle of e_i .

Let $\gamma_1, \dots, \gamma_{l_1+i-1}$ be the generators of $\pi_1(L_i)$, we will suppose that if $j \leq i$, $\gamma_1, \dots, \gamma_{l_1+j-1}$ are generators of $\pi_1(L_j)$. Remark that since we have supposed that (L_2, ∇_{L_2}) is a circle bundle over (L_1, ∇_{L_1}) , we can assume that $h_{x_2}(\gamma_{l_1+1})$ is the identity, that is the generator of the element of $\pi_1(L_2)$ which preserves leaves of the circle bundle.

Let c_1 be the cocycle which defines the isomorphism class of the lagrangian foliation \mathcal{L}_1 on U_1 , for each element γ of $\pi_1(L_1)$, $c_1(\gamma)$ is a germ of a differentiable function defined on \mathbb{R}^{l_1} . Let γ' be an element of $\pi_1(L_2)$ over γ , then $c_2(\gamma')$ is a germ of a differentiable function of \mathbb{R}^{l_1+1} over $c_1(\gamma)$, where c_2 is a classifying cocycle of an element of $\text{symp}(L_2, e_1)$.

Consider $L(h_i)$, the linear holonomy of the affine manifold (L_i, ∇_{L_i}) , and write $\mathbb{R}^{l_1+1} = \mathbb{R}^{l_1} + \mathbb{R}$. For each element $\gamma \in \pi_1(L_2)$, $L(h_2(\gamma))$ depends of $p_2(\gamma)$ and the projection of $L(h_2(\gamma))$ on $\mathbb{R}e_{l_1+1}$ parallel to \mathbb{R}^{l_1} is a cocycle d_2 for the trivial action of $\pi_1(L_2)$ on $\mathbb{R}e_{l_1+1}$.

We will suppose now that the holonomy is linearizable in neighbourhoods compact leaves L_1 and L_2 . We will determine the relation between the cocycle which define elements of $\text{symp}(L_2, e_1)$ and the classifying cocycle of e_1 .

For each element γ of $\pi_1(L_1)$, we consider a differentiable function $c_2(\gamma)$ of \mathbb{R}^{l_1+1} such that $c_2(\gamma)$ project to $c_1(\gamma)$, which means that there exists a function $f_2(\gamma)$ such that

$$c_2(\gamma) = c_1(\gamma)(x_1, \dots, x_{l_1}) + f_2(\gamma)(x_{l_1+1})$$

and for γ_{l_1+1} , we consider an element $c_2(\gamma_{l_1+1})$ which projects to 0, that is a function of x_{l_1+1} the $l_1 + 1$ coordinate in the basis (e_1, \dots, e_{l_1+1}) of \mathbb{R}^{l_1+1} , the fact that the map c_2 is a cocycle implies the following:

$$c_2(\gamma_{l_1+1}\gamma) = c_2(\gamma_{l_1+1}) + c_2(\gamma)$$

For every element γ of $\pi_1(L_2)$,

$$c_2(\gamma\gamma') = c_2(\gamma) + \gamma c_2(\gamma') =$$

$$c_1(p_2(\gamma)) + f_2(\gamma) + p_2(\gamma)c_1(p_2(\gamma)) + f_2(\gamma') \circ d_2(\gamma)$$

This implies that:

$$f_2(\gamma\gamma') = f_2(\gamma) + f_2(\gamma') \circ d_2(\gamma)$$

More generally, the classifying cocycle $c_i(h)$ of an element h of $\text{symp}(L_i)$ is described by a germ of a differentiable function $f_i(h)$ of $\mathbb{R}e_{l_1+i-1}$, such that

$$c_i(h)(\gamma) = c_{i-1}(p_i(\gamma) + f_i(h)(\gamma))$$

which satisfies:

$$f_i(h)(\gamma\gamma') = f_i(h)(\gamma) + f_i(h)(\gamma) \circ d_i(\gamma).$$

Cocycles in this theory are computable when we suppose that the lagrangian foliation is linearizable.

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